On the Central Limit Theorem for the Eigenvalue Counting Function of Wigner and Covariance matrices

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Abstract. *This note presents some central limit theorems for the eigenvalue counting function of Wigner matrices in the form of suitable translations of results by Gustavsson and O'Rourke on the limiting behavior of eigenvalues inside the bulk of the semicircle law for Gaussian matrices. The theorems are then extended to large families of Wigner matrices by the Tao and Vu Four Moment Theorem. Similar results are developed for covariance matrices.*

Recent developments in random matrix theory have concerned the limiting behavior of linear statistics of random matrices when the size n of the matrix goes to infinity (see for example: [4], [11], [17], [2], [15], [14]). In this work, we restrict ourselves to the families of so-called Wigner and covariance matrices.

Wigner matrices are Hermitian or real symmetric matrices M_n such that, if M_n is complex, for $i < j$, the real and imaginary parts of $(M_n)_{ij}$ are iid, with mean 0 and variance $\frac{1}{2}$, $(M_n)_{ii}$ are iid, with mean 0 and variance 1. In the real case, $(M_n)_{ij}$ are iid, with mean 0 and variance 1 and $(M_n)_{ii}$ are iid, with mean 0 and variance 2. In both cases, set $W_n = \frac{1}{\sqrt{n}} M_n$. An important example of Wigner matrices is the case where the entries are Gaussian. If M_n is complex, then it belongs to the so-called Gaussian Unitary Ensemble (GUE). If it is real, it belongs to the Gaussian Orthogonal Ensemble (GOE). In this case, the joint law of the eigenvalues is known, allowing for complete descriptions of their limiting behavior both in the global and local regimes (cf. for example [1]).

Covariance matrices are Hermitian or real symmetric semidefinite matrices $S_{m,n}$ such that $S_{m,n} = \frac{1}{n} X^* X$ where X is a $m \times n$ random complex or real matrix (with $m \geqslant n$) whose entries are iid with mean 0 and variance 1. We only consider here the situation where $\frac{m}{n} \to \gamma \in [1, +\infty)$ as $n \to \infty$. If the entries are Gaussian, then the covariance matrix belongs to the so-called Laguerre Unitary Ensemble (LUE) if it is complex and Laguerre Orthogonal Ensemble (LOE) if it is real. Again in this Gaussian case, the joint law of the eigenvalues is known allowing, as for Wigner matrices, for a complete knowledge of their asymptotics (see for example [3], [5], [13]).

Both W_n and $S_{m,n}$ have n real eigenvalues $\lambda_1, \ldots, \lambda_n$ for which one may investigate the linear statistics

$$N_n[\varphi] = \sum_{j=1}^{n} \varphi(\lambda_j)$$

where $\varphi : \mathbb{R} \to \mathbb{C}$. The classical Wigner theorem states that the empirical distribution $\frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j}$ on the eigenvalues of W_n converges weakly almost surely to the semi-circle law $d\rho_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbb{1}_{[-2,2]}(x)dx$. Consequently, for any bounded continuous function φ,

$$\frac{1}{n}N_n[\varphi] \underset{n\to+\infty}{\to} \int \varphi d\rho_{sc} \quad \text{almost surely.}$$

At the fluctuation level, various results have been obtained in the last decade for different subclasses of Wigner matrices. As usual in random matrix theory, the case of GUE or GOE matrices is easier and was investigated first. For a regular function φ, it has been shown by Johansson (see [12]) that the random variable $N_n^o[\varphi] = N_n[\varphi] - \mathbb{E}[N_n[\varphi]]$ converges in distribution to a Gaussian random variable with mean 0 and variance

$$V^{\beta}_{\text{Gaussian}}[\varphi] = \frac{1}{2\beta\pi^2}\int_{-2}^{2}\int_{-2}^{2}\left(\frac{\varphi(x)-\varphi(y)}{x-y}\right)^2 \frac{4-xy}{\sqrt{4-x^2}\sqrt{4-y^2}}\,dxdy,$$

where $\beta = 1$ if the matrix is from the GOE and $\beta = 2$ if it is from the GUE. Cabanal-Duvillard in [7] proved this theorem using different techniques. It is remarkable that due to the repelling properties of the eigenvalues, no normalization appears in this central limit theorem. Recentely, Lytova and Pastur [14] proved this theorem with weaker assumptions for the smoothness of φ: if φ is continuous and has a bounded derivative, the theorem is true.

The case of covariance matrices is very similar. The Marchenko-Pastur theorem states that the empirical distribution $\frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j}$ on the eigenvalues of $S_{m,n}$ converges weakly almost surely to the Marchenko-Pastur law (with parameter γ) $d\mu_\gamma(x) = \frac{1}{2\pi x}\sqrt{(x-\alpha)(\beta-x)}\mathbb{1}_{[\alpha,\beta]}(x)dx$, where $\alpha = (\sqrt{\gamma}-1)^2$ and $\beta = (\sqrt{\gamma}+1)^2$. Consequently, for any bounded continuous function φ,

$$\frac{1}{n}N_n[\varphi] \underset{n\to+\infty}{\to} \int \varphi d\mu_\gamma \quad \text{almost surely.}$$

At the fluctuation level, Guionnet (cf. [10]) proved that, for $S_{m,n}$ from the LUE and φ a polynomial function, the random variable $N_n^o[\varphi] = N_n[\varphi] - \mathbb{E}[N_n[\varphi]]$ converges in distribution to a Gaussian random variable with mean 0 and variance

$$V^{\beta}_{\text{Laguerre}}[\varphi] = \frac{1}{2\beta\pi^2}\int_{-\alpha}^{\beta}\int_{-\alpha}^{\beta}\left(\frac{\varphi(x)-\varphi(y)}{x-y}\right)^2 \frac{4\gamma-(x-\delta)(y-\delta)}{\sqrt{4\gamma-(x-\delta)^2}\sqrt{4\gamma-(y-\delta)^2}}\,dxdy,$$

where $\beta = 1$ if the matrix is from the LOE and $\beta = 2$ if it is from the LUE, and $\delta = \frac{\alpha+\beta}{2} = 1+\gamma$. Again, Cabanal-Duvillard in [7] proved this theorem using different techniques. Recently, Lytova and Pastur in [14] proved that this result is true for continuous test functions φ with a bounded derivative.

Numerous recent investigations (cf. [2], [4]) have been concerned with the extension of the preceding statements to non-Gaussian Wigner and covariance matrices. More or less, the results are the same but so far stronger smoothness assumptions on φ are required. Various techniques have been developed toward this goal: moment method for polynomial functions φ (see [2]), Stieltjès transform for analytical functions φ (see [4]) and Fourier transforms for essentially $\mathcal{C}^4$ functions φ (see [14]). The latest and perhaps more complete results are due to Lytova and Pastur [14] who proved that, under some suitable assumptions on the distribution of the entries of the Wigner matrix, the smoothness condition $\mathcal{C}^4$ on φ is essentially enough to ensure that $N_n^o[\varphi]$ converges in distribution to a Gaussian random variable with mean 0 and variance $V_{Wig}[\varphi]$ which is the sum of $V^{\beta}_{\text{Gaussian}}[\varphi]$ and a term which is zero in the Gaussian case. In the same article, they proved a similar result for covariance matrices: under the same assumptions, $N^o_n[\varphi]$ converges in distribution to a Gaussian random variable with mean 0 and variance $V_{Cov}[\varphi]$ which is the sum of $V^{\beta}_{\text{Laguerre}}[\varphi]$ and a term which is zero in the Gaussian case. These results are deduced from the Gaussian cases by using an interpolation procedure.

The picture is rather different when φ is not smooth, and much is less actually known in this case. What is best known concerns the case where φ is the characteristic function of an interval I, in which case $N_n[\varphi]$ is the number of eigenvalues falling into the interval I, and which will be denoted by $N_I(W_n)$ or $N_I(S_{m,n})$ throughout this work.

By Wigner's and Marchenko-Pastur's theorems as above, for any interval $I \subset \mathbb{R}$,

$$\frac{1}{n}N_I(W_n) \underset{n\to+\infty}{\to} \rho_{sc}(I) \text{ and } \frac{1}{n}N_I(S_{m,n}) \underset{n\to+\infty}{\to} \mu_\gamma(I) \quad \text{almost surely.}$$

In case of the GUE and the LUE, the eigenvalues form a determinantal point process. This particular structure (cf. [6]) allows for the representation of N_I as a sum of independent Bernoulli random variables with parameters related to the kernel eigenvalues. In particular, this description underlies the following general central limit theorem going back to Costin-Lebowitz and Soshnikov (cf. [8] and [16]).

Theorem 1 (Costin-Lebowitz, Soshnikov)**.** *Let M_n be a GUE matrix. Let I_n be an interval in $\mathbb{R}$. If $\mathrm{Var}(N_{I_n}(M_n)) \to \infty$ as $n \to \infty$, then*

$$\frac{N_{I_n}(M_n) - \mathbb{E}[N_{I_n}(M_n)]}{\sqrt{\mathrm{Var}(N_{I_n}(M_n))}} \to \mathcal{N}(0,1) \tag{1}$$

in distribution.

The same result holds for a LUE matrix $S_{m,n}$.

In order to efficiently use this conclusion, it is of interest to evaluate the order of growth of the variance of $N_I(M_n)$. As a main result, Gustavsson [11], using asymptotics of Hermite orthogonal polynomials, was able to show that, say for an interval I strictly in the bulk $(-2,+2)$ of the semi-circle law, $\mathrm{Var}(N_I(M_n))$ is of the order of $\sqrt{\log n}$ as $n \to \infty$. This behavior is thus in strong contrast with the smooth case (for which no normalization is necessary). On the basis of this result, Gustavsson investigated the (Gaussian) limiting behavior of eigenvalues in the bulk. The main observation in this regard is the link between the k-th eigenvalue λ_k (sorted in nondecreasing order) and the counting function $N_I(W_n)$ of an interval $I = (-\infty, a]$, $a \in \mathbb{R}$, given by

$$N_I(W_n) \geqslant k \quad \text{if and only if} \quad \lambda_k \leqslant a. \tag{2}$$

Gustavsson's results have been extended to the real GOE ensemble by O'Rourke in [15] by means of interlacing formulas (cf. [9]). Using their already famous Four Moment Theorem, Tao and Vu (cf. [18]) were able to extend Gustavsson's theorem to large classes of Wigner Hermitian matrices. As pointed out at the end of O'Rourke's paper [15], the extension also holds for real Wigner matrices.

Su [17] extended Gustavsson's work for LUE matrices and got the same behavior for the variance of $N_I(S_{m,n})$, namely $\sqrt{\log n}$. Similar interlacing results (cf. [9]) yield the same conclusions for LOE matrices. Since Tao and Vu extended their Four Moment Theorem to covariance matrices (cf. [20]), it is then possible to extend Su's central limit theorems to more general covariance matrices.

The purpose of this note is to translate the aforementioned results on the behavior of eigenvalues inside the bulk directly as central limit theorems on the eigenvalue counting function, combining thus the Costin-Lebowitz - Soshnikov theorem with the Tao-Vu Four Moment Theorem. While these statements are implicit in the preceding investigations, we found it interesting and useful to emphasize the conclusions as central limit theorems for the eigenvalue counting function, in particular by comparison with the case of smooth linear statistics as described above. In particular, we express central limit theorems for $N_{[a_n,+\infty)}$, $a_n \to a$, where a is in the bulk of the spectrum and where a_n is close to the edge of the spectrum. The results are presented first, along the lines of Gustavsson [11], for matrices from the GUE, then extended to Wigner Hermitian matrices by the Tao-Vu Four Moment Theorem. Similar results are developed for a finite number of intervals, by means of the corresponding multidimensional central limit theorem. The conclusions are carried over to the real case following O'Rourke's [15] interlacing approach. The results are then presented for LUE matrices, using Su's work [17], and extended to non-Gaussian complex covariance matrices. Following O'Rourke, the results are then extended to LOE matrices and, at last, to real covariance matrices.

Turning to the content of this note, the first section describes the families of Wigner matrices of interest, as well as the Tao-Vu Four Moment Theorem. Section 2 then presents the various central limit theorems for the eigenvalue counting function of Hermitian matrices, both for single or multiple intervals. In Section 3, we formulate the corresponding statements in the real case. In the last section, we present the results for covariance matrices.

1 Notations and definitions

1.1 Wigner matrices

Definitions of Wigner matrices somewhat differ from one paper to another. Here we follow Tao and Vu in [18], in particular for moment assumptions which will be suited to their Four Moment Theorem.

Definition 1. *A Wigner Hermitian matrix of size n is a random Hermitian matrix M_n whose entries ξ_{ij} have the following properties:*

- *For $1 \leqslant i < j \leqslant n$, the real and imaginary parts of ξ_{ij} are iid copies of a real random variable ξ with mean 0 and variance $\frac{1}{2}$.*
- *For $1 \leqslant i \leqslant n$, the entries ξ_{ii} are iid copies of a real random variable $\tilde{\xi}$ with mean 0 and variance 1.*
- *ξ and $\tilde{\xi}$ are independent and have finite moments of high order: there is a constant $C_0 \geqslant 2$ such that*
$$\mathbb{E}[|\xi|^{C_0}] \leqslant C \quad \textit{and} \quad \mathbb{E}[|\tilde{\xi}|^{C_0}] \leqslant C$$
for some constant C.

If ξ and $\tilde{\xi}$ are real Gaussian random variables with mean 0 and variance $\frac{1}{2}$ and 1 respectively, then M_n belongs to the GUE.

A similar definition holds for real Wigner matrices.

Definition 2. *A real Wigner symmetric matrix of size n is a random real symmetric matrix M_n whose entries ξ_{ij} have the following properties:*

- *for $1 \leqslant i < j \leqslant n$, ξ_{ij} are iid copies of a real random variable ξ of mean 0 and variance 1.*
- *for $1 \leqslant i \leqslant n$, ξ_{ii} are iid copies of a real random variable $\tilde{\xi}$ of mean 0 and variance 2.*
- *the entries are independent and have finite moments of high order: there is a constant $C_0 \geqslant 2$ such that*
$$\mathbb{E}[|\xi|^{C_0}] \leqslant C \quad \textit{and} \quad \mathbb{E}[|\tilde{\xi}|^{C_0}] \leqslant C$$
for some constant C.

The GOE is the equivalent of the GUE in the real case, namely a real Wigner symmetric matrix is said to belong to the GOE if its entries are independent Gaussian random variables with mean 0 and variance 1, 2 on the diagonal.

The Gaussian Unitary and Orthogonal Ensembles are specific sets of random matrices for which the eigenvalue density is explicitly known. On this basis, the asymptotic behavior of the eigenvalues, both at the global and local regimes, has been successfully analyzed in the past giving rise to complete and definitive results (cf. for example [1]). Recent investigations have concerned challenging extensions to non-Gaussian Wigner matrices. In this regard, a remarkable breakthrough was achieved by Tao and Vu with their Four Moment Theorem which is a tool allowing the transfer of known results for the GUE or GOE to large classes of Wigner matrices. We present next this main statement following the recent papers [18], [19] and [20].

1.2 Tao and Vu's results

The Tao and Vu Four Moment Theorem indicates that two random matrices whose entries have the same first four moments have very close eigenvalues. Before recalling the precise statement, say that two complex random variables ξ and ξ' match to order k if

$$\mathbb{E}\left[\mathrm{Re}(\xi)^m \mathrm{Im}(\xi)^l\right] = \mathbb{E}\left[\mathrm{Re}(\xi')^m \mathrm{Im}(\xi')^l\right]$$

for all $m, l \geqslant 0$ such that $m + l \leqslant k$.

Theorem 2 (Four Moment Theorem)**.** *There exists a small positive constant c_0 such that, for every $k \geqslant 1$, the following holds. Let $M_n = (\xi_{ij})_{1 \leqslant i,j \leqslant n}$ and $M'_n = (\xi'_{ij})_{1 \leqslant i,j \leqslant n}$ be two random Wigner Hermitian matrices. Assume that, for $1 \leqslant i < j \leqslant n$, ξ_{ij} and ξ'_{ij} match to order 4 and that, for $1 \leqslant i \leqslant n$, ξ_{ii} and ξ'_{ii} match to order 2. Set $A_n = \sqrt{n} M_n$ and $A'_n = \sqrt{n} M'_n$. Let $G : \mathbb{R}^k \to \mathbb{R}$ be a smooth fonction such that:*

$$\forall\ 0 \leqslant j \leqslant 5, \quad \forall x \in \mathbb{R}^k, \quad |\nabla^j G(x)| \leqslant n^{c_0}. \tag{3}$$

Then, for all $1 \leqslant i_1 < i_2 \cdots < i_k \leqslant n$ and for n large enough (depending on k and constants C and C'),

$$|\mathbb{E}[G(\lambda_{i_1}(A_n), \dots, \lambda_{i_k}(A_n))] - \mathbb{E}[G(\lambda_{i_1}(A'_n), \dots, \lambda_{i_k}(A'_n))]| \leqslant n^{-c_0}. \tag{4}$$

This theorem applies to any kind of Wigner matrices. It will be used with one of the two matrices in the GUE giving thus rise to the following corollary.

Corollary 3. *Let M_n be a Wigner Hermitian matrix such that ξ satisfies $\mathbb{E}[\xi^3] = 0$ and $\mathbb{E}[\xi^4] = \frac{3}{4}$. Let M'_n be a matrix from the GUE. Then, with G, A_n, A'_n as in the previous theorem, and n large enough,*

$$|\mathbb{E}[G(\lambda_{i_1}(A_n), \dots, \lambda_{i_k}(A_n))] - \mathbb{E}[G(\lambda_{i_1}(A'_n), \dots, \lambda_{i_k}(A'_n))]| \leqslant n^{-c_0}. \tag{5}$$

For the further purposes, let us briefly illustrate how this result may be used in order to estimate tail probabilities of eigenvalues. Consider matrices satisfying the conditions of Theorem 2. Let $I = [a, b]$, $I^+ = [a - n^{-c_0/10}, b + n^{-c_0/10}]$ and $I^- = [a + n^{-c_0/10}, b - n^{-c_0/10}]$. Take a smooth bump function G such that $G(x) = 1$ if $x \in I$ and $G(x) = 0$ if $x \notin I^+$ and such that G satisfies condition (3). Theorem 2 applies in this setting so that, for every $i \in \{1, \dots, n\}$ (possibly depending on n),

$$|\mathbb{E}[G(\lambda_i(A_n))] - \mathbb{E}[G(\lambda_i(A'_n))]| \leqslant n^{-c_0}.$$

But now, $\mathbb{P}(\lambda_i(A_n) \in I) \leqslant \mathbb{E}[G(\lambda_i(A_n))]$ and $\mathbb{P}(\lambda_i(A'_n) \in I^+) \geqslant \mathbb{E}[G(\lambda_i(A'_n))]$. Therefore, by the triangle inequality,

$$\mathbb{P}(\lambda_i(A_n) \in I) \leqslant \mathbb{P}(\lambda_i(A'_n) \in I^+) + n^{-c_0}.$$

Taking another smooth bump function H such that $H(x) = 1$ if $x \in I^-$ and $H(x) = 0$ if $x \notin I$ and using the same technique yields

$$\mathbb{P}(\lambda_i(A'_n) \in I^-) - n^{-c_0} \leqslant \mathbb{P}(\lambda_i(A_n) \in I).$$

Combining the two preceding inequalities,

$$\mathbb{P}(\lambda_i(A'_n) \in I^-) - n^{-c_0} \leqslant \mathbb{P}(\lambda_i(A_n) \in I) \leqslant \mathbb{P}(\lambda_i(A'_n) \in I^+) + n^{-c_0}. \tag{6}$$

This inequality will be used repeatedly. Combined with the equivalence (2), it will yield significant informations on the eigenvalue counting function. In the next section we thus present the central limit theorems for GUE matrices, and then transfer them, by this tool, to Hermitian Wigner matrices. The real case will be addressed next.

2 Central limit theorems (CLT) for Hermitian matrices

2.1 Infinite intervals

2.1.1 CLT for GUE matrices

We first present Gustavsson's results [11] on the limiting behavior of the expectation and variance of the eigenvalue counting function of the GUE, and then deduce the corresponding central limit theorems through the Costin-Lebowitz - Soshnikov Theorem (Theorem 1). Set $G(t) = \rho_{sc}((-\infty, t]) = \frac{1}{2\pi}\int_{-2}^{t} \sqrt{4 - x^2}\mathbb{1}_{[-2,2]}(x)dx$ for $t \in [-2, 2]$.

Theorem 4. *Let M_n be a GUE matrix.*

- *Let $t = G^{-1}\left(\frac{k}{n}\right)$ with $\frac{k}{n} \to a \in (0, 1)$. The number of eigenvalues of M_n in the interval $I_n = [t\sqrt{n}, +\infty)$ has the following asymptotics:*

$$\mathbb{E}[N_{I_n}(M_n)] = n - k + O\Big(\frac{\log n}{n}\Big). \tag{7}$$

- *The expected number of eigenvalues of M_n in the interval $I_n = [t_n\sqrt{n}, +\infty)$, when $t_n \to 2^-$, is given by:*

$$\mathbb{E}[N_{I_n}(M_n)] = \frac{2}{3\pi}n(2 - t)^{3/2} + O(1). \tag{8}$$

- *Let $\delta > 0$. Assume that t_n satisfies $t_n \in [-2+\delta, 2)$ and $n(2-t_n)^{3/2} \to +\infty$ when $n \to \infty$. Then the variance of the number of eigenvalues of M_n in $I_n = [t_n\sqrt{n}, +\infty)$ satisfies*

$$\mathrm{Var}(N_{I_n}(M_n)) = \frac{1}{2\pi^2} \log[n(2-t_n)^{3/2}](1+\eta(n)), \tag{9}$$

where $\eta(n) \to 0$ as $n \to \infty$.

As announced, together with Theorem 1, we deduce central limit theorems for the eigenvalue counting function of the GUE.

Theorem 5. *Let M_n be a GUE matrix and $W_n = \frac{1}{\sqrt{n}} M_n$ Set $I_n = [a_n, +\infty)$, where $a_n \to a \in (-2, 2)$ when $n \to \infty$. Then*

$$\frac{N_{I_n}(W_n) - n\rho_{sc}([a_n, +\infty))}{\sqrt{\frac{1}{2\pi^2} \log n}} \to \mathcal{N}(0,1), \tag{10}$$

in distribution when n goes to ∞.

The proof is an immediate consequence of Theorem 4 together with the Costin-Lebowitz - Soshnikov Theorem (Theorem 1). Note that the statement holds similarly with $\mathbb{E}[N_{I_n}(W_n)]$ instead of $n\rho_{sc}([a_n, +\infty))$ (and actually the latter is a consequence of the result with $\mathbb{E}[N_{I_n}(W_n)]$).

Theorem 5 concerns intervals in the bulk. When the interval is close to the edge, the second part of Theorem 4 yields the corresponding conclusion.

Theorem 6. *Let M_n be a GUE matrix and $W_n = \frac{1}{\sqrt{n}} M_n$. Let $I_n = [a_n, +\infty)$ where $a_n \to 2^-$ when n goes to infinity. Assume actually that a_n satisfies $a_n \in [-2+\delta, 2)$ and $n(2-a_n)^{3/2} \to +\infty$ when $n \to \infty$. Then, as n goes to infinity,*

$$\frac{N_{I_n}(W_n) - \frac{2}{3\pi} n(2-a_n)^{3/2}}{\sqrt{\frac{1}{2\pi^2} \log\left[n(2-a_n)^{3/2}\right]}} \to \mathcal{N}(0,1), \tag{11}$$

in distribution.

2.1.2 CLT for Wigner matrices

On the basis of the preceding results for GUE matrices, we now deduce the corresponding statements for Hermitian Wigner matrices using the Four Moment Theorem (Theorem 2).

Theorem 7. *Let M_n be a Wigner Hermitian matrix satisfying the hypotheses of Theorem 2 with a GUE matrix M'_n. Set $W_n = \frac{1}{\sqrt{n}} M_n$ as usual. Set $I_n = [a_n, +\infty)$ where $a_n \to a \in (-2, 2)$. Then, as n goes to infinity,*

$$\frac{N_{I_n}(W_n) - n\rho_{sc}([a_n, +\infty))}{\sqrt{\frac{1}{2\pi^2} \log n}} \to \mathcal{N}(0,1) \tag{12}$$

in distribution.

Proof. Let $x \in \mathbb{R}$. We have

$$\mathbb{P}\bigg(\frac{N_{I_n}(W_n) - n\rho_{sc}([a_n, +\infty))}{\sqrt{\frac{1}{2\pi^2}\log n}} \leqslant x\bigg) = \mathbb{P}(N_{I_n}(W_n) \leqslant n - k_n)$$

where $k_n = \big\lceil n\rho_{sc}((-\infty, a_n]) - x\sqrt{\frac{1}{2\pi^2}\log n}\ \big\rceil$. Hence, by (2),

$$\mathbb{P}\bigg(\frac{N_{I_n}(W_n) - n\rho_{sc}([a_n, +\infty))}{\sqrt{\frac{1}{2\pi^2}\log n}} \leqslant x\bigg) = \mathbb{P}(\lambda_{k_n}(M_n) \leqslant a_n\sqrt{n}) = \mathbb{P}(\lambda_{k_n}(A_n) \leqslant a_n n)$$

where $A_n = \sqrt{n}M_n$. Set $A'_n = \sqrt{n}M'_n$. By Theorem 2, more precisely (6),

$$\mathbb{P}\left(\lambda_{k_n}(A'_n) \leqslant a_n n - n^{-c_0/10}\right) - n^{-c_0} \leqslant \mathbb{P}\left(\lambda_{k_n}(A_n) \leqslant a_n n\right),$$

and

$$\mathbb{P}\left(\lambda_{k_n}(A_n) \leqslant a_n n\right) \leqslant \mathbb{P}\left(\lambda_{k_n}(A'_n) \leqslant a_n n + n^{-c_0/10}\right) + n^{-c_0}.$$

Start with the probability on the right of the preceding inequalities (the term n^{-c_0} going to 0 as $n \to \infty$). We have,

$$\begin{aligned}
\mathbb{P}\big(\lambda_{k_n}(A'_n) \leqslant a_n n + n^{-c_0/10}\big) &= \mathbb{P}\big(\lambda_{k_n}(M'_n) \leqslant (a_n + n^{-1-c_0/10})\sqrt{n}\big) \\
&= \mathbb{P}\big(N_{[a_n + n^{-1-c_0/10}, +\infty)}(W'_n) \leqslant n - k_n\big) \\
&= \mathbb{P}\bigg(N_{[a'_n, +\infty)}(W'_n) \leqslant n\rho_{sc}([a_n, +\infty)) + x\sqrt{\frac{1}{2\pi^2}\log n}\bigg)
\end{aligned}$$

where $a'_n = a_n + n^{-1-c_0/10}$. Therefore,

$$\mathbb{P}\big(\lambda_{k_n}(A'_n) \leqslant a_n n + n^{-c_0/10}\big) = \mathbb{P}\bigg(\frac{N_{[a'_n, +\infty[}(W'_n) - n\rho_{sc}([a'_n, +\infty))}{\sqrt{\frac{1}{2\pi^2}\log n}} \leqslant \frac{n\rho_{sc}([a_n, a'_n])}{\sqrt{\frac{1}{2\pi^2}\log n}} + x\bigg).$$

Recall from Theorem 5 that

$$X_n = \frac{N_{[a'_n, +\infty[}(W'_n) - n\rho_{sc}([a'_n, +\infty[)}{\sqrt{\frac{1}{2\pi^2}\log n}} \to \mathcal{N}(0, 1),$$

in distribution as n goes to infinity. Set now $x_n = \frac{n\rho_{sc}([a_n, a'_n])}{\sqrt{\frac{1}{2\pi^2}\log n}} + x$. In order to describe the asymptotic behavior of x_n, observe that

$$\begin{aligned}
\rho_{sc}([a_n, a'_n]) &= G(a'_n) - G(a_n) \\
&= G'(a_n)(a'_n - a_n) + o(a'_n - a_n) \\
&= \frac{1}{2\pi}\sqrt{4 - a_n^2}\, n^{-1-c_0/10} + o(n^{-1-c_0/10}).
\end{aligned}$$

It immediately follows that $x_n \to x$. We are thus left to show that $\mathbb{P}(X_n \leqslant x_n) \to \mathbb{P}(X \leqslant x)$ where $X \sim \mathcal{N}(0, 1)$. To this task, let $\varepsilon > 0$. There exists n_0 such that, for all $n \geqslant n_0$,

$x - \varepsilon \leqslant x_n \leqslant x + \varepsilon$. Then, for all $n \geqslant n_0$, $\mathbb{P}(X_n \leqslant x - \varepsilon) \leqslant \mathbb{P}(X_n \leqslant x_n) \leqslant \mathbb{P}(X_n \leqslant x + \varepsilon)$. Hence

$$\limsup_{n \to +\infty} \mathbb{P}(X_n \leqslant x_n) \leqslant \limsup_{n \to +\infty} \mathbb{P}(X_n \leqslant x + \varepsilon) = \mathbb{P}(X \leqslant x + \varepsilon).$$

and

$$\liminf_{n \to +\infty} \mathbb{P}(X_n \leqslant x_n) \geqslant \liminf_{n \to +\infty} \mathbb{P}(X_n \leqslant x - \varepsilon) = \mathbb{P}(X \leqslant x - \varepsilon).$$

Since the distribution of X is continuous, the conclusion follows.

The same argument works for the lower bound in equation (2.1.2). The proof of the theorem is then easily completed. $\square$

It should be mentioned that, for arbitrary Wigner matrices, since $\mathbb{E}[N_{I_n}(W_n)]$ does not obviously behave like $n\rho_{sc}([a_n, +\infty))$, it is not clear thus whether the statement holds similarly with $\mathbb{E}[N_{I_n}(W_n)]$ instead of $n\rho_{sc}([a_n, +\infty))$.

We next state and prove the corresponding result for intervals close to the edge.

Theorem 8. *Let M_n be a Wigner Hermitian matrix satisfying the hypotheses of Theorem 2 with a GUE matrix M_n'. Set $W_n = \frac{1}{\sqrt{n}} M_n$. Set $I_n = [a_n, +\infty)$, where $a_n \to 2^-$ when n goes to infinity. Then, as $n \to \infty$,*

$$\frac{N_{I_n}(W_n) - \frac{2}{3\pi} n(2 - a_n)^{3/2}}{\sqrt{\frac{1}{2\pi^2} \log[n(2 - a_n)^{3/2}]}} \to \mathcal{N}(0, 1). \tag{13}$$

Proof. Let $x \in \mathbb{R}$.

$$\mathbb{P}\left(\frac{N_{I_n}(W_n) - \frac{2}{3\pi} n(2 - a_n)^{3/2}}{\sqrt{\frac{1}{2\pi^2} \log[n(2 - a_n)^{3/2}]}} \leqslant x \right) = \mathbb{P}(N_{I_n}(W_n) \leqslant k_n),$$

where $k_n = \left\lfloor \frac{2}{3\pi} n(2 - a_n)^{3/2} + x \sqrt{\frac{1}{2\pi^2} \log[n(2 - a_n)^{3/2}]} \right\rfloor$. Then, by (2),

$$\begin{aligned} \mathbb{P}\left(\frac{N_{I_n}(W_n) - \frac{2}{3\pi} n(2 - a_n)^{3/2}}{\sqrt{\frac{1}{2\pi^2} \log[n(2 - a_n)^{3/2}]}} \leqslant x \right) &= \mathbb{P}(\lambda_{n-k_n}(M_n) \leqslant a_n \sqrt{n}) \\ &= \mathbb{P}(\lambda_{n-k_n}(A_n) \leqslant a_n n), \end{aligned}$$

where $A_n = \sqrt{n} M_n$. Set $A_n' = \sqrt{n} M_n'$. Using Theorem 2, more precisely (6),

$$\mathbb{P}\big(\lambda_{n-k_n}(A_n') \leqslant a_n n - n^{-c_0/10}\big) - n^{-c_0} \leqslant \mathbb{P}\big(\lambda_{n-k_n}(A_n) \leqslant a_n n\big),$$

and

$$\mathbb{P}\big(\lambda_{n-k_n}(A_n) \leqslant a_n n\big) \leqslant \mathbb{P}\big(\lambda_{n-k_n}(A_n') \leqslant a_n n + n^{-c_0/10}\big) + n^{-c_0}.$$

Start with the probability on the right of the preceding inequality (the term n^{-c_0} going to 0 as $n \to \infty$).

$$\begin{aligned} \mathbb{P}\big(\lambda_{n-k_n}(A_n') \leqslant a_n n + n^{-c_0/10}\big) &= \mathbb{P}\big(\lambda_{n-k_n}(M_n') \leqslant (a_n + n^{-1-c_0/10})\sqrt{n}\big) \\ &= \mathbb{P}\big(N_{[a_n', +\infty)}(W_n') \leqslant k_n\big), \end{aligned}$$

Then

$$\mathbb{P}\Big(\lambda_{n-k_n}(A'_n) \leqslant a_n n + n^{-c_0/10}\Big) = \mathbb{P}\Big(N_{[a'_n,+\infty)}(W'_n) \leqslant \frac{2}{3\pi} n(2-a_n)^{3/2} + x\sqrt{\frac{1}{2\pi^2}\log[n(2-a_n)^{3/2}]}\Big),$$

where $a'_n = a_n + n^{-1-c_0/10}$. Set

$$X_n = \frac{N_{[a'_n,+\infty)}(W'_n) - \frac{2}{3\pi} n(2-a'_n)^{3/2}}{\sqrt{\frac{1}{2\pi^2}\log[n(2-a'_n)^{3/2}]}}.$$

Then

$$\mathbb{P}\left(\lambda_{n-k_n}(A'_n) \leqslant a_n n + n^{-c_0/10}\right) \;=\; \mathbb{P}\left(X_n \leqslant x_n\right),$$

where

$$x_n = \frac{\frac{2}{3\pi} n\left[(2-a_n)^{3/2} - (2-a'_n)^{3/2}\right]}{\sqrt{\frac{1}{2\pi^2}\log[n(2-a'_n)^{3/2}]}} + x\sqrt{\frac{\log[n(2-a_n)^{3/2}]}{\log[n(2-a'_n)^{3/2}]}}.$$

We need to know if Theorem 6 apply. We must have $a'_n \to 2^-$ and $n(2-a'_n)^{3/2} \to +\infty$ when n goes to infinity. First, we can see that $a_n \to 2$. Suppose now that n is such that $a'_n - 2 > 0$. Then $a_n - 2 + n^{-1-c_0/10} > 0$. Then $2 - a_n < n^{-1-c_0/10}$. As $2 - a_n > 0$ for n large enough, $n(2-a_n)^{3/2} < n n^{-\frac{3c_0}{20}-\frac{3}{2}}$. And we get $n(2-a_n)^{3/2} < n^{-\frac{3c_0}{20}-\frac{1}{2}}$. But $n(2-a_n)^{3/2}$ goes to infinity and $n^{-\frac{3c_0}{20}-\frac{1}{2}}$ goes to 0, which means that this situation is impossible if n is large enough. Then, for n large enough, $2 - a'_n > 0$ and $a'_n \to 2^-$. Now, turn to the second condition. Namely,

$$\begin{aligned}
(2-a'_n)^{3/2} &= (2-a_n-n^{-1-c_0/10})^{3/2}\\
&= (2-a_n)^{3/2}\Big(1 - \frac{n^{-1-c_0/10}}{2-a_n}\Big)^{3/2}\\
&= (2-a_n)^{3/2}\Bigg(1 - \frac{3}{2}\frac{n^{-1-c_0/10}}{2-a_n} + o\Big(\frac{n^{-1-c_0/10}}{2-a_n}\Big)\Bigg),
\end{aligned}$$

as, when $n \to \infty$,

$$\frac{n^{-1-c_0/10}}{2-a_n} = \frac{n^{-\frac{c_0}{10}-\frac{1}{3}}}{[n(2-a_n)^{3/2}]^{2/3}} \to 0.$$

Then

$$n(2-a'_n)^{3/2} \;=\; n(2-a_n)^{3/2}\Bigg(1 - \frac{3}{2}\frac{n^{-1-c_0/10}}{2-a_n} + o\Big(\frac{n^{-1-c_0/10}}{2-a_n}\Big)\Bigg).$$

But $n(2-a_n)^{3/2}$ goes to $+\infty$ when $n \to \infty$ and $\frac{n^{-1-c_0/10}}{2-a_n} \to 0$. Then $n(2-a'_n)^{3/2} \to \infty$, and Theorem 6 apply. Consequently, when n goes to infinity,

$$\frac{N_{[a'_n,+\infty)}(W'_n) - \frac{2}{3\pi} n(2-a'_n)^{3/2}}{\sqrt{\frac{1}{2\pi^2}\log[n(2-a'_n)^{3/2}]}} \to \mathcal{N}(0,1),$$

in distribution.

The argument will be completed provided $x_n \to x$. Using the preceding,

$$n\big[(2-a_n)^{3/2} - (2-a'_n)^{3/2}\big] = \frac{3}{2} n^{-c_0/10}(2-a_n)^{1/2} + o\big(n^{-c_0/10}\big) \to 0.$$

Furthermore, $\sqrt{\frac{1}{2\pi^2}\log[n(2-a'_n)^{3/2}]} \to \infty$. Therefore,

$$\frac{\frac{2}{3\pi} n \left[(2-a_n)^{3/2} - (2-a'_n)^{3/2}\right]}{\sqrt{\frac{1}{2\pi^2}\log[n(2-a'_n)^{3/2}]}} \to 0.$$

Moreover,

$$\begin{aligned}
\frac{\log[n(2-a_n)^{3/2}]}{\log[n(2-a'_n)^{3/2}]} &= \frac{\log[n(2-a_n)^{3/2}]}{\log\left(n(2-a_n)^{3/2}[1-\frac{n^{-1-c_0/10}}{2-a_n}]^{3/2}\right)} \\
&= \frac{\log[n(2-a_n)^{3/2}]}{\log[n(2-a_n)^{3/2}] + \frac{3}{2}\log[1-\frac{n^{-1-c_0/10}}{2-a_n}]} \\
&\to 1.
\end{aligned}$$

Hence $x_n = x + o(1)$. The proof of the theorem may then be concluded as the one of Theorem 7. $\square$

2.2 Finite intervals

In this section, we investigate the corresponding results for finite intervals in the bulk. Namely, we would like to study $N_{[a,b]}(W_n)$ (for $-2 < a, b < 2$). To this task, write $N_{[a,b]}(W_n) = N_{[a,+\infty)}(W_n) - N_{[b,+\infty)}(W_n)$ so that we are led to study the couple $(N_{[a,+\infty)}(W_n), N_{[b,+\infty)}(W_n))$. For more complicated sets, such as $[a,b] \cup [c,+\infty)$ with a,b,c in the bulk, we need to study the relations between three or more such quantities. In this subsection, we thus investigate the m-tuple $(N_{[a^1,+\infty)}(W_n), \ldots, N_{[a^m,+\infty)}(W_n))$ for which we establish a multidimensional central limit theorem.

2.2.1 CLT for GUE matrices

As for infinite intervals, we start with GUE matrices.

Theorem 9. *Let M_n be a GUE matrix and set $W_n = \frac{1}{\sqrt{n}} M_n$. Let m be a fixed integer. Let $a_n^i \to a^i$ for all $i \in \{1,\ldots,m\}$, with $-2 < a^1 < a^2 < \cdots < a^m < 2$. Set, for all $i \in \{1,\ldots,m\}$,*

$$X_{a^i}(W_n) = \frac{N_{[a_n^i,+\infty)}(W_n) - n\rho_{sc}([a_n^i,+\infty))}{\sqrt{\frac{1}{2\pi^2}\log n}}.$$

Then, as n goes to infinity,

$$\big((X_{a^1}(W_n), \ldots, X_{a^m}(W_n)\big) \to \mathcal{N}(0, I_m),$$

in distribution, where I_m is the identity matrix of size m.

Proof. In order to simplify notations, we will denote $N_I = N_I(W_n)$ throughout the proof, for all interval I in $\mathbb{R}$.

By means of the multidimensional version of the Costin-Lebowitz - Soshnikov theorem, Gustavsson [11] showed that, for all $(\beta_1, \dots, \beta_m) \in \mathbb{R}^m$,

$$\frac{\sum_{j=1}^m \beta_j N_{[a_n^j, a_n^{j+1})} - \mathbb{E}\big[\sum_{j=1}^m \beta_j N_{[a_n^j, a_n^{j+1})}\big]}{\sqrt{\mathrm{Var}\,\big(\sum_{j=1}^m \beta_j N_{[a_n^j, a_n^{j+1})}\big)}} \underset{n\to\infty}{\to} \mathcal{N}(0,1),$$

in distribution, with $a_n^{m+1} = \infty$. But, for all $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$,

$$\frac{\sum_{k=1}^m \alpha_k N_{[a_n^k, +\infty)} - \mathbb{E}\big[\sum_{k=1}^m \alpha_k N_{[a_n^k, \infty)}\big]}{\sqrt{\mathrm{Var}\,\big(\sum_{k=1}^m \alpha_k N_{[a_n^k, \infty)}\big)}} = \frac{\sum_{j=1}^m \beta_j N_{[a_n^j, a_n^{j+1})} - \mathbb{E}\big[\sum_{j=1}^m \beta_j N_{[a_n^j, a_n^{j+1})}\big]}{\sqrt{\mathrm{Var}\,\big(\sum_{j=1}^m \beta_j N_{[a_n^j, a_n^{j+1})}\big)}},$$

with $\beta_j = \sum_{k=1}^j \alpha_k$. Therefore,

$$\frac{\sum_{k=1}^m \alpha_k N_{[a_n^k, +\infty)} - \mathbb{E}\big[\sum_{k=1}^m \alpha_k N_{[a_n^k, +\infty)}\big]}{\sqrt{\mathrm{Var}\,\big(\sum_{k=1}^m \alpha_k N_{[a_n^k, +\infty)}\big)}} \underset{n\to\infty}{\to} \mathcal{N}(0,1),$$

in distribution.

Set now $Y_k = N_{[a_n^k, +\infty)}$. To see whether $\big(\frac{Y_k - \mathbb{E}[Y_k]}{\sqrt{\mathrm{Var}\,Y_k}}\big)_{1\leqslant k\leqslant m}$ converges in distribution, we proceed as follows. Gustavsson showed that the covariance matrix of $\big(\frac{Y_k - \mathbb{E}[Y_k]}{\sqrt{\mathrm{Var}\,Y_k}}\big)_{1\leqslant k\leqslant m}$ has limit $\Sigma = I_m$ as n goes to infinity. Let then $(\beta_1, \dots, \beta_m)$ be in $\mathbb{R}^m$. Set $\alpha_k = \frac{\beta_k}{\sqrt{\mathrm{Var}(Y_k)}}$. In distribution,

$$\frac{\sum_{k=1}^m \alpha_k Y_k - \mathbb{E}\big[\sum_{k=1}^m \alpha_k Y_k\big]}{\sqrt{\mathrm{Var}\,\big(\sum_{k=1}^m \alpha_k Y_k\big)}} \underset{n\to\infty}{\to} \mathcal{N}(0,1).$$

But

$$\mathrm{Var}\left(\sum_{k=1}^m \alpha_k Y_k\right) = \sum_{k,l=1}^m \frac{\beta_k \beta_l}{\sqrt{\mathrm{Var}\,Y_k\,\mathrm{Var}\,Y_l}} \mathrm{Cov}(Y_k, Y_l) \underset{n\to\infty}{\to} \sum_{k,l=1}^m \beta_k \beta_l \Sigma_{kl},$$

Then, using Slutsky's lemma,

$$\sum_{k=1}^m \beta_k \left(\frac{Y_k}{\sqrt{\mathrm{Var}\,Y_k}} - \frac{\mathbb{E}[Y_k]}{\sqrt{\mathrm{Var}\,Y_k}}\right) \to \mathcal{N}(0, {}^t\beta\Sigma\beta).$$

Since this is true for every $\beta \in \mathbb{R}^m$ and as $\Sigma = I_m$,

$$\left(\frac{Y_k - \mathbb{E}[Y_k]}{\sqrt{\mathrm{Var}\,Y_k}}\right)_{1\leqslant k\leqslant m} \underset{n\to\infty}{\to} \mathcal{N}(0, I_m)$$

in distribution. Using the asymptotics for the mean and the variance that Gustavsson calculated in the one-dimensional case (cf. Theorem 4), we easily conclude to the announced convergence

$$(X_{a^1}(W_n), \dots, X_{a^m}(W_n)) \underset{n\to\infty}{\to} \mathcal{N}(0, I_m),$$

in distribution. The proof is thus complete. $\square$

2.2.2 CLT for Wigner matrices

We use the same techniques as in the one-dimensional case in order to extend the preceding theorem to Wigner Hermitian matrices.

Theorem 10. *Let M_n be a Wigner Hermitian matrix satisfying the hypotheses of Theorem 2 with a GUE matrix M_n'. Using the same notations as before, as $n \to \infty$,*

$$(X_{a^1}(W_n), \dots, X_{a^m}(W_n)) \to \mathcal{N}(0, I_m)$$

in distribution, where I_m is the identity matrix of size m.

Proof. Let $x_1, \dots, x_m$ be in $\mathbb{R}$.

$$\begin{aligned} & \mathbb{P}\left(X_{a^1}(W_n) \leqslant x_1, \dots, X_{a^m}(W_n) \leqslant x_m\right) \\ = \; & \mathbb{P}\left(N_{[a_n^i, +\infty)}(W_n) \leqslant n\rho_{sc}([a_n^i, +\infty)) + x_i\sqrt{\tfrac{1}{2\pi^2}\log n},\ 1 \leqslant i \leqslant m\right) \\ = \; & \mathbb{P}\left(N_{[a_n^i, +\infty)}(W_n) \leqslant n - k_i,\ 1 \leqslant i \leqslant m\right), \end{aligned}$$

where $k_i = \left\lceil n\rho_{sc}((-\infty, a_n^i]) - x_i\sqrt{\frac{1}{2\pi^2}\log n}\ \right\rceil$. Then

$$\begin{aligned} \mathbb{P}\left(X_{a^1}(W_n) \leqslant x_1, \dots, X_{a^m}(W_n) \leqslant x_m\right) &= \mathbb{P}\left(\lambda_{k_i}(M_n) \leqslant a_n^i\sqrt{n},\ 1 \leqslant i \leqslant m\right) \\ &= \mathbb{P}\left(\lambda_{k_i}(A_n) \leqslant a_n^i n,\ 1 \leqslant i \leqslant m\right), \end{aligned}$$

where $A_n = \sqrt{n}M_n$. Set $A_n' = \sqrt{n}M_n'$. Now, use Theorem 2 with m eigenvalues. What is important to note here is that the inequalities do not depend on which eigenvalues are chosen. They only depend on the number m of eigenvalues. Applying thus the same arguments as for one eigenvalue, we get

$$\mathbb{P}\left(\lambda_{k_i}(A_n') \in I_i^-,\ 1 \leqslant i \leqslant m\right) - o_m(n^{-c_0}) \leqslant \mathbb{P}\left(\lambda_{k_i}(A_n) \in I_i,\ 1 \leqslant i \leqslant m\right), \tag{14}$$

and

$$\mathbb{P}\left(\lambda_{k_i}(A_n) \in I_i,\ 1 \leqslant i \leqslant m\right) \leqslant \mathbb{P}\left(\lambda_{k_i}(A_n') \in I_i^+,\ 1 \leqslant i \leqslant m\right) + o_m(n^{-c_0}), \tag{15}$$

where I_i are intervals, I_i^+ and I_i^- are intervals deduced from I_i by adding $\pm n^{-c_0/10}$. Thus,

$$\mathbb{P}\left(\lambda_{k_i}(A_n') \leqslant a_n^i n - n^{-c_0/10},\ 1 \leqslant i \leqslant m\right) - o_m(n^{-c_0}) \leqslant \mathbb{P}\left(\lambda_{k_i}(A_n) \leqslant a_n^i n,\ 1 \leqslant i \leqslant m\right),$$

and

$$\mathbb{P}\left(\lambda_{k_i}(A_n) \leqslant a_n^i n,\ 1 \leqslant i \leqslant m\right) \leqslant \mathbb{P}\left(\lambda_{k_i}(A_n') \leqslant a_n^i n + n^{-c_0/10},\ 1 \leqslant i \leqslant m\right) + o_m(n^{-c_0}).$$

Consider the probability on the right in the preceding inequality (the term $o_m(n^{-c_0})$ going to 0 when $n \to \infty$). We have,

$$\begin{aligned} \mathbb{P}\left(\lambda_{k_i}(A_n') \leqslant a_n^i n + n^{-c_0/10},\ 1 \leqslant i \leqslant m\right) &= \mathbb{P}\left(\lambda_{k_i}(M_n') \leqslant (a_n^i + n^{-1-c_0/10})\sqrt{n},\ 1 \leqslant i \leqslant m\right) \\ &= \mathbb{P}\left(N_{[(a_n^i)', +\infty)}(W_n') \leqslant n - k_i,\ 1 \leqslant i \leqslant m\right) \end{aligned}$$

where $(a_n^i)' = a_n^i + n^{-1-c_0/10}$. Then

$$\mathbb{P}\left(\lambda_{k_i}(A_n') \leqslant a_n^i n + n^{-c_0/10},\ 1 \leqslant i \leqslant m\right) = \mathbb{P}\left(X_{(a^i)'}(W_n') \leqslant (x_n^i)',\ 1 \leqslant i \leqslant m\right)$$

with

$$X_{(a^i)'}(W_n') = \frac{N_{[(a_n^i)',+\infty)}(W_n') - \rho_{sc}([(a_n^i)',+\infty))}{\sqrt{\frac{1}{2\pi^2}\log n}}$$

and

$$(x_n^i)' = \frac{n\rho_{sc}([a_n^i,(a_n^i)'])}{\sqrt{\frac{1}{2\pi^2}\log n}} + x_i.$$

We know that $(a_n^i)' \to a^i$. Then, Theorem 10 apply and we have

$$\big(X_{(a^1)'}(W_n'),\dots,X_{(a^m)'}(W_n')\big) \to \mathcal{N}(0,I_m).$$

We are left with the asymptotics of $(x_n^i)'$. To this task, note that

$$\rho_{sc}([a_n^i,(a_n^i)']) = \int_{a_n^i}^{(a_n^i)'} \frac{1}{2\pi}\sqrt{4-x^2}dx = \frac{1}{2\pi}\sqrt{4-(a_n^i)^2}((a_n^i)'-a_n^i) + o((a_n^i)'-a_n^i).$$

Then

$$\begin{aligned}
\frac{n\rho_{sc}([a_n^i,(a_n^i)'])}{\sqrt{\frac{1}{2\pi^2}\log n}} &= \frac{n}{\sqrt{\frac{1}{2\pi^2}\log n}}\left(\frac{1}{2\pi}\sqrt{4-(a_n^i)^2}n^{-1-c_0/10} + o(n^{-1-c_0/10})\right)\\
&= \frac{1}{\sqrt{\frac{1}{2\pi^2}\log n}}\left(\frac{1}{2\pi}\sqrt{4-(a_n^i)^2}n^{-c_0/10} + o(n^{-c_0/10})\right)\\
&\to 0,
\end{aligned}$$

when n goes to infinity. Therefore $x_n^i \to x^i$. Since $\big(X_{(a^1)'}(W_n'),\dots,X_{(a^m)'}(W_n')\big) \to (Y_1,\dots,Y_m)$ where $Y=(Y_1,\dots,Y_m)\sim\mathcal{N}(0,I_m)$, as in the one-dimensional case, it easily follows that

$$\mathbb{P}\big(X_{(a^i)'}(W_n) \leqslant (x_n^i)',\ 1\leqslant i\leqslant m\big) \to \mathbb{P}\big(Y_i \leqslant x_i,\ 1\leqslant i\leqslant m\big).$$

Together with the same considerations for the lower bounds, the conclusion follows. The theorem is thus established. ☐

On the basis of the preceding result, we conclude to the announced central limit theorem for the number of eigenvalues in a finite interval $[a_n,b_n]$.

Theorem 11. *Let M_n be a Wigner Hermitian matrix satisfying the hypotheses of Theorem 2 with a GUE matrix M_n'. Set $W_n = \frac{1}{\sqrt{n}}M_n$. Let $I_n=[a_n,b_n]$ where $a_n\to a$, $b_n\to b$ and $-2<a<b<2$. Then, as n goes to infinity,*

$$\frac{N_{I_n}(W_n) - n\rho_{sc}([a_n,b_n])}{\sqrt{\frac{1}{2\pi^2}\log n}} \to \mathcal{N}(0,2) \tag{16}$$

in distribution.

Proof. From the preceding theorem, we know that

$$\left(\frac{N_{[a_n,+\infty)}(W_n) - n\rho_{sc}([a_n,+\infty))}{\sqrt{\frac{1}{2\pi^2}\log n}}, \frac{N_{[b_n,+\infty)}(W_n) - n\rho_{sc}([b_n,+\infty))}{\sqrt{\frac{1}{2\pi^2}\log n}}\right) \to \mathcal{N}(0,I_2).$$

But

$$N_{[a_n,b_n]}(W_n) = N_{[a_n,+\infty)}(W_n) - N_{[b_n,+\infty)}(W_n) = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} N_{[a_n,+\infty)}(W_n) \\ N_{[b_n,+\infty)}(W_n) \end{pmatrix}.$$

Therefore,

$$\frac{N_{[a_n,+\infty)}(W_n) - n\rho_{sc}([a_n,+\infty))}{\sqrt{\frac{1}{2\pi^2}\log n}} - \frac{N_{[b_n,+\infty)}(W_n) - n\rho_{sc}([b_n,+\infty))}{\sqrt{\frac{1}{2\pi^2}\log n}}$$

behaves asymptotically like a Gaussian random variable with mean 0 and variance $\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix}^T = 2$, as $n \to \infty$. Then

$$\frac{N_{[a_n,b_n]}(W_n) - n\rho_{sc}([a_n,b_n])}{\sqrt{\frac{1}{2\pi^2}\log n}} \to \mathcal{N}(0,2),$$

which concludes the proof. $\square$

Remark: The result on the m-tuple

$$\left(\frac{N_{[a_n^1,+\infty)}(W_n) - n\rho_{sc}([a_n^1,+\infty))}{\sqrt{\frac{1}{2\pi^2}\log n}}, \dots, \frac{N_{[a_n^m,+\infty)}(W_n) - n\rho_{sc}([a_n^m,+\infty))}{\sqrt{\frac{1}{2\pi^2}\log n}}\right),$$

with $a_n^i \to a^i$ and $-2 < a^1 < \cdots < a^m < 2$, yields further central limit theorems. For example, we can deduce a central limit theorem for $N_{[a_n,b_n]\cup[c_n,+\infty)}(W_n)$ where $a_n \to a$, $b_n \to b$, $c_n \to c$ and $-2 < a < b < c < 2$. Indeed,

$$N_{[a_n,b_n]\cup[c_n,+\infty)}(W_n) = N_{[a_n,+\infty)}(W_n) - N_{[b_n,+\infty)}(W_n) + N_{[c_n,+\infty)}(W_n).$$

And then, when n goes to infinity,

$$\frac{N_{[a_n,b_n]\cup[c_n,+\infty)}(W_n) - n\rho_{sc}([a_n,b_n]) - n\rho_{sc}([c_n,+\infty))}{\sqrt{\frac{1}{2\pi^2}\log n}} \to \mathcal{N}(0,3),$$

in distribution.

3 Real matrices

In this section, we briefly indicate how the preceding results for Hermitian random matrices may be stated similarly for real Wigner symmetric matrices. To this task, we follow the same scheme of proof, relying in particular on the Tao and Vu Four Moment Theorem (Theorem 2) which also holds in the real case (cf. [15]). The main issue is actually to establish first the conclusions for the GOE. This has been suitably developed by O'Rourke in [15] by means of interlacing formulas (cf. [9]).

3.1 Links between the GOE and the GUE

We first recall O'Rourke's [15] conclusions for the GOE relying on the following interlacing formula of Forrester and Rains [9].

Theorem 12 (Forrester-Rains)**.** *The following relation holds between matrix ensembles:*

$$\mathrm{GUE}_n = \mathrm{even}(\mathrm{GOE}_n \cup \mathrm{GOE}_{n+1}). \tag{17}$$

This statement can be interpreted in the following way. Take two independent matrices from the GOE, one of size n and the other of size $n+1$. If we surimperpose the $2n+1$ eigenvalues on the real line and then take the n even ones, they have the same distribution as the eigenvalues of a $n \times n$ matrix from the GUE.

Now, if I_n is an interval of $\mathbb{R}$,

$$N_{I_n}(M_n^{\mathbb{C}}) = \frac{1}{2}\big(N_{I_n}(M_n^{\mathbb{R}}) + N_{I_n}(M_{n+1}^{\mathbb{R}}) + \xi_n(I_n)\big),$$

where $M_n^{\mathbb{C}}$ is a $n \times n$ GUE matrix, $M_n^{\mathbb{R}}$ a $n \times n$ GOE matrix and $\xi_n(I_n)$ takes values in $\{-1, 0, 1\}$. The following interlacing property will then lead to the expected conclusions.

Theorem 13 (Cauchy's interlacing theorem)**.** *If A is a Hermitian matrix and B is a principle submatrix of A, then the eigenvalues of B interlace with the eigenvalues of A. In other words, if $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ are the eigenvalues of A and $\mu_1 \leqslant \cdots \leqslant \mu_{n-1}$ the eigenvalues of B, then*

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \mu_{n-1} \leqslant \lambda_n.$$

This theorem enables us to make a link between the eigenvalues of a $(n+1) \times (n+1)$ GOE matrix and a $n \times n$ one: $N_{I_n}(M_n^{\mathbb{R}}) = N_{I_n}(M_{n+1}^{\mathbb{R}}) + \xi'_n(I_n)$ where $\xi'_n(I_n)$ takes values in $\{-1, 0, 1\}$. In particular,

$$N_{I_n}(M_n^{\mathbb{C}}) = N_{I_n}(M_n^{\mathbb{R}}) + \zeta_n(I_n), \tag{18}$$

where $\zeta_n(I_n)$ takes values in $\left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$.

3.2 Infinite intervals

3.2.1 CLT for GOE matrices

On the basis of the preceding tools, O'Rourke extends in [15] Gustavsson's results from the GUE to the GOE. The first statement is the analogue of the Costin-Lebowitz - Soshnikov Theorem (Theorem 1).

Theorem 14. *Let $M_n^{\mathbb{R}}$ be a GOE matrix and $M_n^{\mathbb{C}}$ a GUE matrix. Let I_n be an interval in $\mathbb{R}$. If $\mathrm{Var}(N_{I_n}(M_n^{\mathbb{C}})) \to +\infty$ when $n \to \infty$,*

$$\frac{N_{I_n}(M_n^{\mathbb{R}}) - \mathbb{E}[N_{I_n}(M_n^{\mathbb{R}})]}{\sqrt{\mathrm{Var}(N_{I_n}(M_n^{\mathbb{C}}))}} \to \mathcal{N}(0, 2). \tag{19}$$

In a second step, using the interlacing principle, O'Rourke develops estimates for the mean of the number of eigenvalues in an interval for GOE matrices.

Lemma 15. *Let $M_n^{\mathbb{R}}$ be a GOE matrix.*

- *Let $t = G^{-1}\left(\frac{k}{n}\right)$ with $\frac{k}{n} \to a \in (0,1)$. Let $I_n = [t\sqrt{n}, +\infty)$. Then*
$$\mathbb{E}[N_{I_n}(M_n^{\mathbb{R}})] = n - k + O(1). \tag{20}$$
- *Let $I_n = [t_n\sqrt{n}, +\infty)$, where $t_n \to 2^-$. Then*
$$\mathbb{E}[N_{I_n}(M_n^{\mathbb{R}})] = \frac{2}{3\pi}n(2-t)^{3/2} + O(1). \tag{21}$$

Following the same line of arguments as in the complex case, we may thus formulate the corresponding central limit theorems for the eigenvalue counting function of the GOE.

Theorem 16. *Let $M_n^{\mathbb{R}}$ be a GOE matrix and set $W_n^{\mathbb{R}} = \frac{1}{\sqrt{n}}M_n^{\mathbb{R}}$. Let $I_n = [a_n, +\infty)$, where $a_n \to a \in (-2,2)$ when $n \to +\infty$. Then, as n goes to infinity,*
$$\frac{N_{I_n}(W_n^{\mathbb{R}}) - n\rho_{sc}([a_n, +\infty))}{\sqrt{\frac{1}{2\pi^2}\log n}} \to \mathcal{N}(0,2) \tag{22}$$
in distribution.

Similar results hold for intervals close to the edge, as in the complex case. The corresponding statement is presented for general Wigner matrices in the next sub-section.

3.2.2 CLT for real Wigner matrices

In this section, we state the central limit theorems for real symmetric Wigner matrices, as a consequence of the preceding statements and the Four Moment Theorem (Theorem 2), which is completely similar in the real case. The proofs are exactly the same as in the complex case.

Theorem 17. *Let $M_n^{\mathbb{R}}$ be a Wigner symmetric matrix satisfying the hypotheses of Theorem 2 with a GOE matrix $(M_n^{\mathbb{R}})'$. Set $W_n^{\mathbb{R}} = \frac{1}{\sqrt{n}}M_n^{\mathbb{R}}$. Set $I_n = [a_n, +\infty)$, where $a_n \to a \in (-2,2)$. Then,*
$$\frac{N_{I_n}(W_n^{\mathbb{R}}) - n\rho_{sc}([a_n, +\infty))}{\sqrt{\frac{1}{2\pi^2}\log n}} \to \mathcal{N}(0,2). \tag{23}$$

Theorem 18. *Let $M_n^{\mathbb{R}}$ be a Wigner symmetric matrix satisfying the hypotheses of Theorem 2 with a GOE matrix $(M_n^{\mathbb{R}})'$. Set $W_n^{\mathbb{R}} = \frac{1}{\sqrt{n}}M_n^{\mathbb{R}}$. Set $I_n = [a_n, +\infty)$, where $a_n \to 2^-$ when n goes to infinity. Assume that a_n satisfies $a_n \in [-2+\delta, 2)$ and $n(2-a_n)^{3/2} \to +\infty$ when $n \to \infty$. Then*
$$\frac{N_{I_n}(W_n^{\mathbb{R}}) - \frac{2}{3\pi}n(2-a_n)^{3/2}}{\sqrt{\frac{1}{2\pi^2}\log[n(2-a_n)^{3/2}]}} \to \mathcal{N}(0,2). \tag{24}$$

3.3 Finite intervals

As in the complex case, we turn now to finite intervals. Following exactly the same scheme, we get similar results.

3.3.1 CLT for GOE matrices

The next statement may be found in [15].

Theorem 19. *Let $M_n^{\mathbb{R}}$ be a GOE matrix and set $W_n^{\mathbb{R}} = \frac{1}{\sqrt{n}} M_n^{\mathbb{R}}$. Let m be a fixed integer. and let $a_n^i \to a^i$ for all $i \in \{1, \dots, m\}$, with $-2 < a^1 < a^2 < \dots < a^m < 2$. Set*

$$X_{a^i}(W_n^{\mathbb{R}}) = \frac{N_{[a_n^i, +\infty)}(W_n^{\mathbb{R}}) - n\rho_{sc}([a_n^i, +\infty))}{\sqrt{\frac{1}{2\pi^2} \log n}}$$

for all $i \in \{1, \dots, m\}$. Then, as n goes to infinity,

$$\big(X_{a^1}(W_n^{\mathbb{R}}), \dots, X_{a^m}(W_n^{\mathbb{R}})\big) \to \mathcal{N}(0, 2I_m),$$

in distribution, where I_m is the identity matrix of size m.

3.3.2 CLT for finite intervals

Using the same techniques as in the complex case, this theorem can be extended to real Wigner symmetric matrices.

4 Covariance matrices

In this section, we briefly present the analogous results for covariance matrices. We rely here on Su's work [17] who developed for the LUE the results corresponding to those of Gustavsson in case of the GUE. We then make use of the Tao and Vu Four Moment Theorem for general covariance matrices [20]. We first recall below precise definitions of the covariance matrices under investigation and Su's contributions.

4.1 Complex covariance matrices

As for Wigner matrices, we add to the usual definition of a covariance matrix an assumption on moments, in order to apply Tao and Vu's Four Moment Theorem.

Definition 3. *Let n and m be integers such that $m \geqslant n$ and $\lim_{n\to\infty} \frac{m}{n} = \gamma \in [1, +\infty)$. Let X be a random $m \times n$ matrix with entries ζ_{ij} such that:*

- $\mathrm{Re}(\zeta_{ij})$ *and* $\mathrm{Im}(\zeta_{ij})$ *have mean* 0 *and variance* $\frac{1}{2}$.
- ζ_{ij} *are independent and identically distributed.*
- *there exists $C_0 \geqslant 2$ and $C \geqslant 0$ (independent of n and m) such that* $\sup_{ij} \mathbb{E}[|\zeta_{ij}|^{C_0}] \leqslant C$.

Then $S_{m,n} = \frac{1}{n} X^* X$ is called a covariance matrix.

$S_{m,n}$ is Hermitian and positive semidefinite with rank at most n. Hence it has at most n non zero eigenvalues, which are real and nonnegative. Denote them by $0 \leqslant \lambda_1(S_{m,n}) \leqslant \dots \leqslant \lambda_n(S_{m,n})$.

An important example of covariance matrices is the case where the entries are Gaussian. These matrices form the so-called Laguerre Unitary Ensemble (LUE). In this case, the distribution of the eigenvalues of $S_{m,n}$ can be explicitely computed. Similarly to the GUE case, it is then possible to compute various local statistics (see for example [3], [5], [13]). In particular, Su [17] was able to estimate the mean and the variance of the number of eigenvalues of $S_{m,n}$ in a given interval. We recall below some of his main conclusions.

Set $\mu_{m,n}(x) = \frac{1}{2\pi x}\sqrt{(x-\alpha_{m,n})(\beta_{m,n}-x)}\mathbb{1}_{[\alpha_{m,n},\beta_{m,n}]}(x)$, where $\alpha_{m,n} = (\sqrt{\frac{m}{n}}-1)^2$ and $\beta_{m,n} = (\sqrt{\frac{m}{n}}+1)^2$. And, if $t \in [\alpha_{m,n}, \beta_{m,n}]$, set $H(t) = \int_{\alpha_{m,n}}^{t} \mu_{m,n}(x)dx$.

Theorem 20. *Let $S_{m,n}$ be a LUE matrix.*

- *Let $t = H^{-1}\left(\frac{k}{n}\right)$ with $\frac{k}{n} \to a \in (0,1)$. The number of eigenvalues of $S_{m,n}$ in the interval $I_n = [t, +\infty)$ has the following asymptotics:*
$$\mathbb{E}[N_{I_n}(S_{m,n})] = n - k + O(1). \tag{25}$$
- *The expected number of eigenvalues of $S_{m,n}$ in the interval $I_n = [t_n, +\infty)$, when $\beta_{m,n} - t_n \to 0^+$ and $n(\beta_{m,n} - t_n)^{3/2} \geqslant C$ for some $C > 0$, is given by:*
$$\mathbb{E}[N_{I_n}(S_{m,n})] = \frac{\sqrt{\beta_{m,n}-\alpha_{m,n}}}{3\pi\beta_{n,m}} n(\beta_{m,n}-t)^{3/2}(1+o(1)). \tag{26}$$
- *Let $\delta > 0$. Assume that $t_n \leqslant \beta_{m,n} - \delta$ for some $\delta > 0$. Then the variance of the number of eigenvalues of $S_{m,n}$ in $I_n = [t_n, +\infty)$ satisfies*
$$\mathrm{Var}(N_{I_n}(S_{m,n})) = \frac{1}{2\pi^2}\log n(1+o(1)). \tag{27}$$
- *Assume that t_n is such that $\beta_{m,n} - t_n \to 0^+$ and $n(\beta_{m,n}-t_n)^{3/2} \geqslant C$ for some $C > 0$. Then the variance of the number of eigenvalues of $S_{m,n}$ in $I_n = [t_n, +\infty)$ satisfies*
$$\mathrm{Var}(N_{I_n}(S_{m,n})) = \frac{1}{2\pi^2}\log[n(\beta_{m,n}-t_n)^{3/2}](1+o(1)). \tag{28}$$

Arguing as in the preceding sections, together with these asymptotics and the Costin-Lebowitz - Soshnikov Theorem 1, the following central limit theorems may be achieved.

Theorem 21. *Let $S_{m,n}$ be a LUE matrix. Set $I_n = [t_n, +\infty)$, where $t_n \to t \in (\alpha, \beta)$ when $n \to \infty$. Then*
$$\frac{N_{I_n}(S_{m,n}) - n\int_{t_n}^{\beta_{m,n}} \mu_{m,n}(x)dx}{\sqrt{\frac{1}{2\pi^2}\log n}} \to \mathcal{N}(0,1), \tag{29}$$
in distribution when n goes to ∞.

Theorem 22. *Let $S_{m,n}$ be a LUE matrix. Let $I_n = [t_n, +\infty)$ where $\beta_{m,n} - t_n \to 0^+$ when n goes to infinity. Assume actually that t_n satisfies $n(\beta_{m,n}-t_n)^{3/2} \to \infty$ when $n \to \infty$. Then, as n goes to infinity,*
$$\frac{N_{I_n}(S_{m,n}) - \frac{\sqrt{\beta_{m,n}-\alpha_{m,n}}}{3\pi\beta_{m,n}} n(\beta_{m,n}-t_n)^{3/2}}{\sqrt{\frac{1}{2\pi^2}\log\left[n(\beta_{m,n}-t_n)^{3/2}\right]}} \to \mathcal{N}(0,1), \tag{30}$$
in distribution.

As was done for Wigner matrices, one can extend these theorems to more general covariance matrices, following exactly the same scheme. Namely, Tao and Vu extended their Four Moment Theorem to the case of covariance matrices in [20]. Using it in the same way as for Wigner matrices, similar comparison properties may be obtained in the form for example of

$$\mathbb{P}(n\lambda_i(S'_{m,n}) \in I^-) - n^{-c_0} \leqslant \mathbb{P}(n\lambda_i(S_{m,n}) \in I) \leqslant \mathbb{P}(n\lambda_i(S'_{m,n}) \in I^+) + n^{-c_0}, \tag{31}$$

where $S_{m,n}$ is a covariance matrix, $S'_{m,n}$ is a Laguerre matrix, $I = [a, b]$, $I^+ = [a - n^{-c_0/10}, b + n^{-c_0/10}]$ and $I^- = [a + n^{-c_0/10}, b - n^{-c_0/10}]$. The preceding theorems are then established exactly as in the Wigner case.

4.2 Real covariance matrices

Real covariance matrices are defined similarly to complex ones. The special case where the entries are Gaussian random variables is called the Laguerre Orthogonal Ensemble (LOE). As in the case of Wigner matrices, there is a link between the eigenvalues of real covariance matrices and those of complex covariance matrices expressed by

$$\mathrm{LUE}_{m,n} = \mathrm{even}(\mathrm{LOE}_{m,n} \cup \mathrm{LOE}_{m+1,n+1}) \tag{32}$$

(cf. [9]). The Cauchy interlacing Theorem 13 (used twice in this case) then indicates that

$$N_{I_n}(S^{\mathbb{C}}_{m,n}) = N_{I_n}(S^{\mathbb{R}}_{m+1,n+1}) + \xi_{m,n}(I_n) \tag{33}$$

where $\xi_{m,n}(I_n)$ takes values in $\{-2, -1, 0, 1, 2\}$, $S^{\mathbb{C}}_{m,n}$ and $S^{\mathbb{R}}_{m+1,n+1}$ are independent. Slightly modifying then the proof by O'Rourke in the Wigner case yields the following statement.

Theorem 23. *Let $S^{\mathbb{C}}_{m,n}$ be a LUE matrix and let $S^{\mathbb{R}}_{m,n}$ be a LOE matrix. Let I_n be an interval of $\mathbb{R}$. If $\mathrm{Var}\, N_{I_n}(S^{\mathbb{C}}_{m,n}) \to \infty$ when $n \to \infty$, then*

$$\frac{N_{I_n}(S^{\mathbb{R}}_{m,n}) - \mathbb{E}[N_{I_n}(S^{\mathbb{R}}_{m,n})]}{\sqrt{\mathrm{Var}(N_{I_n}(S^{\mathbb{C}}_{m,n}))}} \to \mathcal{N}(0, 2).$$

Note that (33) yields $\mathbb{E}[N_{I_n}(S^{\mathbb{R}}_{m,n})] = \mathbb{E}[N_{I_n}(S^{\mathbb{C}}_{m,n})] + O(1)$. Together with Su's Theorem 20 for the mean and the variance of $N_{I_n}(S^{\mathbb{C}}_{m,n})$, we then conclude to similar central limit theorems for intervals in the bulk and near the edge.

On the basis of the Tao and Vu Four Moment Theorem in the real case, the conclusions may be extended to large families of non-Gaussian real covariance matrices.

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